Effective Computation in the Derived Category of Coherent Sheaves in Oscar

SIAM 2025, Madison, Wisconsin, USA

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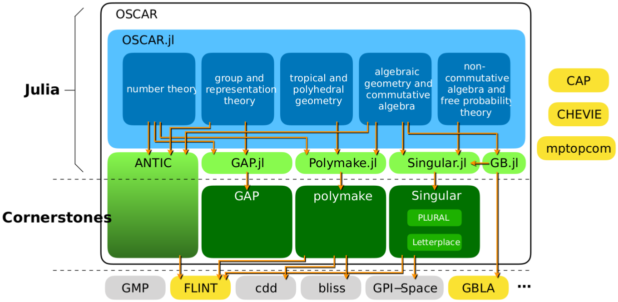


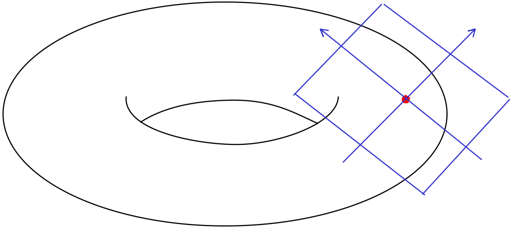
Together with J. Böhm, S. Brandhorst, W. Decker, and A. Frühbis-Krüger we develop the frameworks for Algebraic Geometry and Commutative Algebra within the computer algebra system Oscar [25].

Varieties

$$X = \{x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$$

Coherent sheaves

$$\mathcal{F} = \mathcal{T}_X,$$
$$\mathcal{F}^\vee = \Omega^1_X = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$$

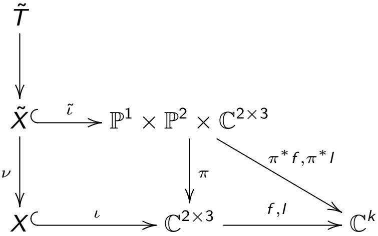


$$\dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \Omega^1_X) = g(X) = \#\text{ holes in } X$$

Many of the most fundamental invariants of algebraic varieties can be expressed as *cohomology groups* of coherent sheaves $H^i(X, \mathcal{F})$.

A more involved example [27]: Suppose $X = \{\varphi \in \mathbb{C}^{2\times 3} : \operatorname{rank} \varphi < 2\}$ is a *generic determinantal variety* and $f\colon \mathbb{C}^{2\times 3} \to \mathbb{C}^k$ a function with an *isolated singularity* on X in the stratified sense.

Let $\nu\colon \tilde{X} \to X$ be the Nash transform for the top-dimensional stratum of X.



We wish to compute the *Milnor number* $\mu(4; f) = \sum_{j=1}^{k} \lambda_j(4; f, l)$ associated to the top-dimensional stratum of X with

$$\lambda_j(4; f, l) = \chi\left(R\nu_*\begin{pmatrix} \mathcal{K}osz(\nu^*(f_1, \dots, f_{j-1}, l_{j+1}, \dots, l_k) \\ \otimes \\ \mathcal{E}NC(\nu^*(\mathrm{d}f_1, \dots, \mathrm{d}f_j, \mathrm{d}l_{j+1}, \dots, \mathrm{d}l_k) \end{pmatrix}\right) - \chi(\dots)$$

for some sufficiently generic linear map $l\colon \mathbb{C}^{2\times 3} \to \mathbb{C}^k$.

A more involved example (continued): This computation requires

- complexes $C^\bullet$ of multigraded modules M over the ring

$$S = (\underbrace{\mathbb{C}[x_{i,j} : i = 1, 2,\ j = 1, 2, 3]}_{\mathbb{C}^{2\times 3}})[\underbrace{u_0, u_1}_{\mathbb{P}^1}, \underbrace{v_0, v_1, v_2}_{\mathbb{P}^2}]$$

- Cartan-Eilenberg resolutions, Koszul complexes, Eagon-Northcott complexes, tensor products of complexes, induced homs, ...
- computation of the direct image of a *complex* of sheaves $\mathcal{F}^\bullet$ associated to a complex of graded modules $\mathcal{F}^\bullet = \widetilde{C}^\bullet$ along the projection

$$\pi \colon \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C}^{2\times 3} \to \mathbb{C}^{2\times 3}.$$

- in particular, this calls for an implementation of the *functoriality*

$$\underbrace{\varphi \colon \mathcal{F} \to \mathcal{G}}_{\in \mathrm{Hom}_{Coh(\dots)}(\mathcal{F},\mathcal{G})} \quad \text{given by} \quad \underbrace{\Phi \colon M \to N}_{\in \mathrm{Hom}_S(M,N)}$$

$$\downarrow$$

$$\underbrace{R\pi_*(\varphi) \colon R\pi_*\mathcal{F} \to R\pi_*\mathcal{G}}_{\in \mathrm{Hom}_{C(\mathbb{C}[x]-\mathrm{Mod})}(\dots,\dots)}$$

by Decker and Eisenbud [14], and "Relative Beilinson monad and Direct Image for Families of coherent sheaves" by Eisenbud and Schreyer [17].

Theoretical foundations: Bernshtein-Gel'fand-Gel'fand [7] and Beilinson [6].

Package: "BGG: Bernstein-Gel'fand-Gel'fand correspondence" [1] in Macaulay2 [22] and "sheafcoh.lib" in Singular [15].

Capabilities: Suppose $S = R[x_0, \dots, x_n]$ is the homogeneous coordinate ring for $\mathbb{P}^n_R$ with R some commutative Noetherian ring.

- When $R = k$ is a field, compute the numbers $h^i(\mathbb{P}^n, \mathcal{F})$ of a sheaf $\mathcal{F} = \widetilde{M}$ on $\mathbb{P}^n_k$ for a field k.
- Let R be a polynomial ring and $\pi\colon \mathbb{P}^n_R \to \operatorname{Spec} R$ the projection. Compute the complex $R\pi_*(\mathcal{F}) \in D^b(\operatorname{Spec} R)$ for a sheaf $\mathcal{F} = \widetilde{M}$ for a finitely generated S-module M.
- Compute induced maps in cohomology for a morphism $\varphi\colon \bigoplus_{i=0}^r S[-d_i] \to \bigoplus_{j=0}^s S[-e_j]$.

Desirable additional functionality:

- Compute the induced maps on the direct images for a morphism of *arbitrary* graded S-modules $\varphi\colon M \to N$ and vice versa for complexes thereof.

by Eisenbud, Erman, Schreyer [16], [18].

Software: “TateOnProducts” [19] in Macualay2 and “tateProdCplxNegGrad_lib” in Singular.

Capabilities: Suppose $S = k[x_{i,j} : i = 1, \dots, n, j = 0, \dots, r_n]$ is the homogeneous coordinate ring of a product of projective spaces $\mathbb{P} = \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$ over a field k and $\mathcal{F} = \widetilde{M}$ is a coherent sheaf on $\mathbb{P}$ for some (multi-) graded S-module M.

- Compute the direct image $R\rho_*(\mathcal{F})$ for the projection $\rho\colon \mathbb{P} \to \mathbb{P}^{r_{j_1}} \times \cdots \times \mathbb{P}^{r_{j_p}}$ to any subset $0 < j_1 < \cdots < j_p \leq n$ of the factors of $\mathbb{P}$.
- Suppose $\mathcal{F}$ is supported on a reduced subscheme $X \subset \mathbb{P}$ and we have a morphism $\varphi\colon X \to \mathbb{P}^m$. Compute the direct image $R\varphi_*(\mathcal{F})$.

Desirable additional functionality:

- Extend the domain from graded S-modules to *complexes* of graded S-modules $\varphi\colon M^\bullet \to N^\bullet$ and their sheafifications.
- Compute the induced maps

$$R\rho_*(\tilde{\varphi})\colon R\rho_*\mathcal{F}^\bullet \to \mathcal{R}\rho_* G^\bullet$$

Fairly recently, Brown and Erman published a method to compute sheaf cohomology on toric varieties via a BGG correspondence [11], [10].

This has been integrated in the Macaulay2 package "MultigradedBGG" by Banks et. al [2].

Blumenhagen, Jurke, Rahn, and Roschy [8], [9], Roschy and Rahn [26], and Shin-Yao [24]. Widely accepted in the Physics community as the fastest way to compute cohomology of toric line bundles.

Software: High performance C++ library published on GitHub [12]; interfacing possible e.g. from Macaulay2 or Oscar.

Capabilities: Given a compact toric variety X over $\mathbb{Q}$ and a toric line bundle $\mathcal{L}$, compute the *numbers* $h^i(X, \mathcal{L})$.

Theoretical foundation: The algorithm is based on *Čech-cohomology* and a sophisticated count of *rationoms*, using the Stanley Reisner ideal of X.

Desirable additional functionality: It would be great to have the functoriality for

$$\varphi\colon \mathcal{L} \to \mathcal{L}' \qquad \rightsquigarrow \qquad H^i(\varphi)\colon H^i(X, \mathcal{L}) \to H^i(X, \mathcal{L}').$$

Software and theoretical foundations for computational homological algebra by Barakat et. al. [5], [4], [3].

Software: See the GitHub of the CAP project or the julia package CAPAndHomalg.jl.

Scope: Formalize the categorical language for computations in homological algebra and in particular sheaves. Capabilities to provide custom compiled "core engines" to carry out specific computational tasks.

This has been part of OSCAR in an earlier stage; eventually we will look into utilizing some aspects again!

Let $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n_k$ with maximal ideal $\mathfrak{m}$ and $\mathcal{F} = \widetilde{M}$ a coherent sheaf on $\mathbb{P}^n$ for some graded S-module M.

We have [23]

$$H^i(\mathbb{P}^n, \mathcal{F}) \cong H^{i+1}_{\mathfrak{m}}(M)_0 \cong \lim_{k\to\infty} \operatorname{Ext}^{i+1}_S(S/\mathfrak{m}^k, M)_0$$

for $i > 0$ and a short exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M)_0 \longrightarrow M_0 \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{F})_0 \longrightarrow H^1_{\mathfrak{m}}(M)_0 \longrightarrow 0$$

for $H^0(\mathbb{P}^n, \mathcal{F})$. Moreover,

$$\lim_{k\to\infty} \operatorname{Ext}^i_S(\mathfrak{m}^k, M) \cong \lim_{k\to\infty} H^i(\operatorname{Kosz}^*(x_0^k, \ldots, x_n^k; M)) \cong H^i(\check{C}^\bullet(\mathfrak{U}; M))$$

for the standard covering $\mathfrak{U} = \bigcup_{i=0}^n \{x_i \neq 0\}$; see e.g. [23, Theorem 2.3].

Note: These modules are not finitely generated over S!

The methods can be extended to work for modules over the Cox ring of a toric variety [13] and for the case where k is not a field but a Noetherian commutative ring.

In OSCAR we have support for

- ((multi-) graded) polynomial rings, quotient rings, and various localizations
- ((multi-) graded) finitely generated modules over such rings
- complexes, double complexes, and, more generally, hypercomplexes of arbitrary dimension over such modules
- additional functionality for complexes such as $\mathrm{Hom}(-,-)$, $-\otimes-$, forming Koszul- and Eagon-Northcott complexes, passing to Cartan-Eilenberg resolutions, forming total complexes, taking strands of a given degree, pruning complexes, ...
- the functorial properties of all these constructions

All of the homological algebra is implemented in a *lazy* way. We write as much *generic* code as possible – greatly influenced by the HomAlg project.

The various high performant backends like Singular [15], Polymake [21], Nemo [20], etc. provide us with reasonable speed for many computations.

Spectral sequences in Čech cohomology [28]

Suppose $\pi\colon \mathbb{P} = \mathbb{P}^{r_1}_R \times_R \cdots \times_R \mathbb{P}^{r_n}_R \to \operatorname{Spec} R$ is a product of projective spaces over $\operatorname{Spec} R$; let

$$S = R[x_{i,j} : i \in \{1, \ldots, n\}, j \in \{0, \ldots, r_i\}], \quad \deg(x_{i,j}) = e_i \in \bigoplus_{i=1}^{n} \mathbb{Z}e_i \cong \mathbb{Z}^n$$

be its homogeneous coordinate ring. Suppose $(M^\bullet, \varphi^\bullet)$ is a suitably bounded complex of finitely generated S-modules. Passing to a free resolution of this complex we may assume

$$M^i = \bigoplus_{j=1}^{b_i} S[-d_{i,j}]$$

to be free for some shifts $d_{i,j} \in \mathbb{Z}^n$. Now

$$R\pi_*(\widetilde{M}^\bullet) = \operatorname{Tot}\left(\varinjlim_k \operatorname{Hom}_S\left(F_k^\bullet, M^\bullet\right)\right)_0 = \varinjlim_k \operatorname{Tot}\left(\operatorname{Hom}_S\left(F_k^\bullet, M^\bullet\right)\right)_0$$

where $\varprojlim F_k^\bullet$ is the *inverse* limit given by resolutions of powers of the irrelevant ideal

$$\mathfrak{m} = \langle x_{j_1} \ldots x_{j_n} : 0 \le j_k \le r_k \rangle \subset S.$$

Spectral sequences in Čech cohomology [28]

We obtain a direct system of double complexes

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
 & & \mathrm{Hom}_S(F^\bullet_{k-1}, M^\bullet)_0 \\
 & \swarrow & \downarrow \\
\lim_{k\to\infty} \mathrm{Hom}_S\left(F^\bullet_k, M^\bullet\right)_0 & \longleftarrow & \mathrm{Hom}_S(F^\bullet_k, M^\bullet)_0 \\
 & \nwarrow & \downarrow \\
 & & \mathrm{Hom}_S(F^\bullet_{k+1}, M^\bullet)_0 \\
 & & \downarrow \\
 & & \vdots
\end{array}
$$

where we have data types for the terms on the right hand side.

In theory, there exists $k_0 \in \mathbb{N}$ so that for $k \geq k_0$ one has

$$R\pi_*(\widetilde{M}^\bullet) \cong_{\mathrm{qis}} \mathrm{Tot}\left(\mathrm{Hom}_S\left(F^\bullet_k, M^\bullet\right)\right)_0 .$$

Spectral sequences in Čech cohomology [28]

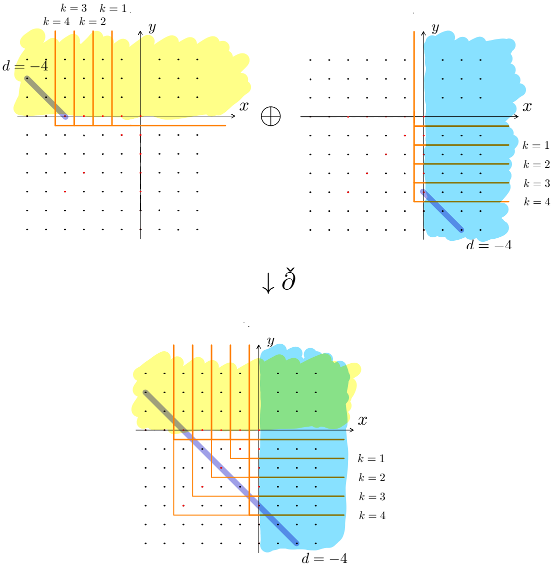
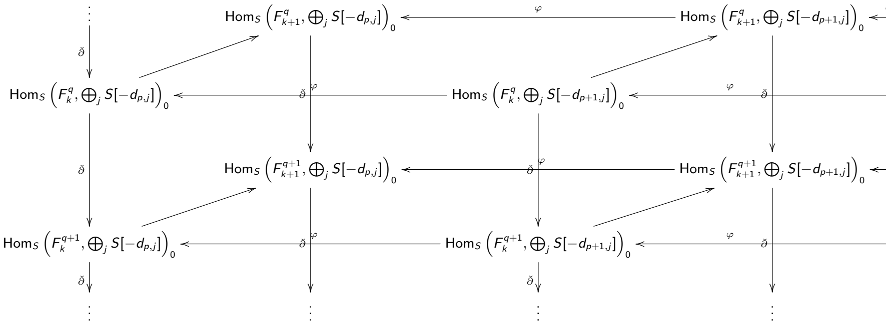


Figure: The Čech complex for $\mathcal{O}_{\mathbb{P}^1}(-4)$

Spectral sequences in Čech cohomology [28]

Spelling out the terms of $\lim_{k\to\infty} \mathrm{Hom}_S(F_k^\bullet, M^\bullet)$:

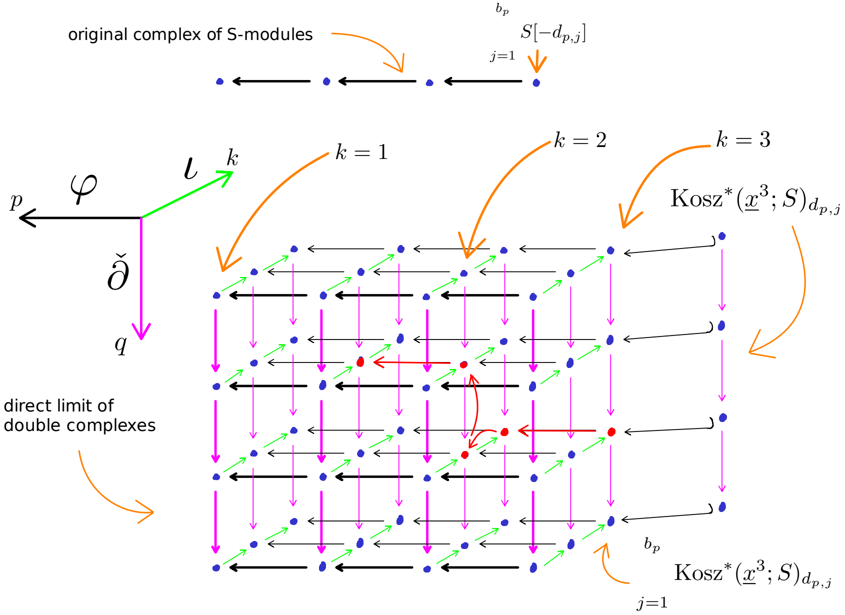


Does this form. . .

- . . . a limit of double complexes of homs of direct sums?
- . . . a double complex of limits of homs of direct sums?
- . . . a double complex of direct sums of limits of hom-modules?

In theory there is no difference, but in practice there is!

Spectral sequences in Čech cohomology [28]



Problem: Computing the direct image for the previous example from a direct limit of double complexes takes ≥ 1 month of computation time.

Observe: There is a massive redundancy in the columns of this limit of double complexes!

$$\lim_{k\to\infty} \mathrm{Hom}_S\left(F_k^\bullet, \bigoplus_j S[-d_{q,j}]\right)_0 \cong \bigoplus_j \Big(\underbrace{\lim_{k\to\infty} \mathrm{Hom}_S\left(F_k^\bullet, S\right)}_{\text{one single limit}}\Big)_{-d_{q,j}}$$

and the Čech maps respect these direct sums.

Idea: Cache the constituents of this single limit and its strands in a context object and recombine the spectral sequence

$$E_1^{p,q} = \bigoplus_j H^q\left(\lim_{k\to\infty} \mathrm{Hom}_S(F_k^\bullet, M^p)\right)_{-d_{p,j}} \Rightarrow H^{p+q}\left(\lim_{k\to\infty} \mathrm{Tot}\left(\mathrm{Hom}_S(F_k^\bullet, M^\bullet)\right)_0\right)$$

from that.

Result: We can compute the spectral sequence for the direct image complex in the previous example in ≤ 2 hours on a laptop!
of OSCAR.

Conclusions and Outlook

- Čech cohomology is not dead!
- Instead of computing higher direct images, it is usually faster to compute a spectral sequence converging to its homology.
- Implementation of functoriality is crucial!
- We have the foundations laid out in OSCAR; how about extending these ideas to. . .
  - . . . cohomology of coherent sheaves on toric varieties?
  - . . . direct images under toric morphisms?
  - . . . using BGG machinery in conjunction with caching objects?

Feel free to reach out to us on GitHub or Slack:

https://www.oscar-system.org/!

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